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Baryon masses at second order in large-N chiral perturbation theory

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Abstract

We consider flavor breaking in the the octet and decuplet baryon masses at second order in large-N chiral perturbation theory, where N is the number of QCD colors. We assume that $1/N \sim 1/N_F \sim m_s/\Lambda \gg m_{u,d}/\Lambda, \alpha_{\rm EM}$, where N_F is the number of light quark flavors, and $m_{u,d,s}/\Lambda$ are the parameters controlling $SU(N_F)$ flavor breaking in chiral perturbation theory. We consistently include non-analytic contributions to the baryon masses at orders $m_q^{3/2}$, $m_q^2 \ln m_q$, and $(m_q \ln m_q)/N$. The $m_q^{3/2}$ corrections are small for the relations that follow from $SU(N_F)$ symmetry alone, but the corrections to the large-N relations are large and have the wrong sign. Chiral power-counting and large-N consistency allow a 2-loop contribution at order $m_q^2 \ln m_q$, and a non-trivial explicit calculation is required to show that this contribution vanishes. At second order in the expansion, there are eight relations that are non-trivial consequences of the $1/N$ expansion, all of which are well satisfied within the experimental errors. The average deviation at this order is 7 MeV for the $\Delta I = 0$ mass differences and 0.35 MeV for the $\Delta I \neq 0$ mass differences, consistent with the expectation that the error is of order $1/N^2 \sim 10\%$.

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I. INTRODUCTION

In this paper, we analyze the octet and decuplet baryon masses to second order in a simultaneous chiral and $1/N$ expansion, where N is the number of QCD colors. The $1/N$ expansion for baryons has a rich structure and significant predictive power: at leading order in the $1/N$ expansion, static baryon matrix elements satisfy $SU(2N_F)$ spin-flavor relations (where N_F is the number of flavors) [1,2] and the $1/N$ corrections to these relations are highly constrained [3–6]. In order to compare the predictions of the $1/N$ expansion with experiment at subleading order, we must consider both $1/N$ corrections and explicit breaking of $SU(N_F)$ flavor symmetry due to quark masses and electromagnetism. We will use the formalism of refs. [6,7] that takes $N_F \sim N \gg 1$ and makes use of an explicit effective lagrangian that keeps the $SU(N_F)$ flavor symmetry manifest order-by-order in the $1/N$ expansion. In order to determine which terms in the double expansion to keep, we will expand assuming that

$$\frac{1}{N} \sim \frac{1}{N_F} \sim \frac{m_s}{\Lambda} \gg \frac{m_{u,d}}{\Lambda} \gg \alpha_{\rm EM}, \tag{1.1}$$

where Λ is the chiral expansion parameter and $\alpha_{\rm EM}$ is the electromagnetic coupling.[1] The baryon magnetic moments were analyzed in the same expansion in ref. [8] and found to be in excellent agreement with experiment. Different $1/N$ expansions for various baryon observables have also been considered in refs. [4,9–11]. In the concluding section we will briefly compare our results with those of ref. [10], which also considers baryon masses.

One important feature of the present work is that we consistently include the chiral loop contributions in our expansion. The leading non-analytic contributions to the baryon mass differences are order $m_q^{3/2}$, $(m_q \ln m_q)/N$, and $m_q^2 \ln m_q$. Power-counting and large-N consistency arguments allow a 2-loop contribution at order $m_q^2 \ln m_q$, and a non-trivial explicit calculation is required to see that such a contribution does not appear. The order $m_q^{3/2}$ corrections are calculable, and the result is that they are small for the relations that follow from $SU(N_F)$ symmetry alone, but the corrections to the large-N relations are large and have the wrong sign. While this may indicate that the expansion parameters are not sufficiently small in nature to believe this expansion, we note that there are higher-order effects that are expected to substantially reduce these corrections. Also, the $m_q^{3/2}$ corrections can be cancelled at higher orders, giving results that agree well with experiment. The coefficients of the $(m_q \ln m_q)/N$ and $m_q^2 \ln m_q$ corrections are not calculable in terms of measured couplings, and we have included them as arbitrary parameters.

Our final results include corrections up to order m_s^2 and m_s/N for the $\Delta I = 0$ mass differences; and order $(m_d - m_u)m_s$, $(m_d - m_u)/N$, and $\alpha_{\rm EM}$ for the $\Delta I \neq 0$ mass differences. At this order there are eight relations which are non-trivial consequences of the $1/N$ expansion. (That is, they do not follow from flavor symmetry considerations alone.) These relations agree well with experiment, and the remaining deviations are consistent with the expectation that the leading corrections are order $1/N^2$.

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[1] See section IIC for a discussion of the last inequality.

This paper is organized as follows. In section 2, we review the expansion used in this paper. In section 3, we present our results, and section 4 contains our conclusions. The details of a 2-loop computation are contained in an appendix.

II. THE EXPANSION

The $1/N$ expansion for baryons has a good deal of predictive power even at subleading orders, but some aspects of the expansion are rather subtle. In this section, we review the main ingredients of this expansion.

A. Baryon quantum numbers

The $1/N$ expansion makes sense only for baryons with spin $J \sim 1$, since baryons with spin $J \sim N$ have width of order N. One conceptual subtlety in the $1/N$ expansion for baryons is that even for fixed $N_F \geq 3$ the number of baryon states with given spin $J \sim 1$ grows with N. When $N_F = 2$, the baryons have quantum numbers $I = J = \frac{1}{2}, \frac{3}{2}, \ldots$, and it is clear that one should identify the states of lowest spin and isospin with the corresponding baryon states at $N = 3$. When $N_F \geq 3$, the situation is more complicated. The Young tableaux for the $SU(N_F)$ representation of spin J baryons is shown in fig. 1; for $N > 3$, this representation contains many states that may be identified with a given baryon state in the world with $N = 3$. Choosing any subset of the baryon states that exist for $N > 3$ to represent the baryons at $N = 3$ breaks the flavor symmetry explicitly. The "extra" states that appear for $N > 3$ are important for computing chiral loops, since they can appear as intermediate states; the contributions of all baryon states are required in order to maintain flavor symmetry and large-N consistency.

A simple way to handle this situation was pointed out in refs. [5,6]. The $1/N$ expansion can be carried out without selecting any subset of the baryon states for $N > 3$ by writing the matrix elements in terms of few-body operators in a spin–flavor Fock space that describes the baryon quantum numbers. In this formalism, the coefficient of an r-body operator is at most $1/N^{r-1}$, and there is a simple classification of the operators that allows us to determine the N- and N_F-dependence of the matrix elements for arbitrary baryon states [6]. For operators with the flavor numbers of a product of $SU(N_F)$ adjoints, any operator can be written as a product of 1-body operators of the form

$$\{X\Gamma\} \equiv a^\dagger_{a\alpha} X^a{}_b \Gamma^\alpha{}_\beta a^{b\beta}, \tag{2.1}$$

where $a^\dagger$ (a) is a creation (annihilation) operator in the spin–flavor Fock space, X is a flavor matrix and Γ is a spin matrix (either 1 or σ^j), with flavor and Lorentz indices contracted in all possible ways. We keep a given operator if its matrix element in *any* baryon state with $J \sim 1$ (for $N_F \geq 3$) is as large as the order to which we are working. It is not hard to see that the largest matrix elements are

$$\{X\Gamma\} \lesssim \begin{cases} 1 & \text{if } X = 1,\ \Gamma = \sigma^j, \\ N & \text{otherwise.} \end{cases} \tag{2.2}$$

Since the operators have definite $SU(N_F)$ quantum numbers, this procedure keeps the flavor symmetry manifest for arbitrary N.

B. Large N_F

In ref. [7] it was shown that the large-N counting rules for baryons are unaffected when the number of flavors is taken to be large, *i.e.* $N_F \sim N$. From the point of view of the quark and gluon degrees of freedom, the reason for this is that the large-N counting rules arise from the suppression of many-quark interactions, and this suppression is unaffected by the presence of quark loops. This is to be contrasted with the large-N predictions for mesons, which generally rely on the suppression of quark loops and therefore are valid only in the limit $N_F \ll N$.

We will assume that $N_F \sim N \gg 1$. This seems sensible, since $N = N_F = 3$ in nature. In order to carry out this expansion, we must decide how to extrapolate flavor breaking (the quark masses and charges) to a world with $N_F > 3$. Clearly, there are infinitely many choices, and our approach will be similar in spirit to the way we handle the additional baryon states that occur: we set up the expansion to be independent of the details of the extrapolation.

We therefore consider extrapolations with arbitrary numbers of individual quark flavors $N_{u,d,s}$ with $N_u + N_d + N_s = N_F$. There are extrapolations where $N_q \sim N_F$ $(q = u, d, s)$, but there are also extrapolations where *e.g.* $N_s \sim 1$ and $N_{u,d} \sim N_F$. We will evaluate the matrix elements for arbitrary N_q, and keep any operator that is as large as the order to which we are working on *any* baryon state for *any* extrapolation. Thus for example, we keep terms of order $1/N_q$ as well as N_q/N.

C. The double expansion

The predictions of the large-N limit for baryons can be summarized by stating that baryon matrix elements obey $SU(2N_F)$ spin–flavor relations in this limit. In order to consider corrections to this limit, we must take into account both the fact that N is finite, and the fact that the $SU(N_F)$ flavor symmetry is broken by quark masses and electromagnetism. We therefore carry out a simultaneous expansion in $1/N$ and flavor breaking. In order to do this, we must decide where to truncate the expansion in the small parameters $1/N \sim 1/N_F$, $m_{u,d,s}/\Lambda$, and α_{EM}.

We will expand the $\Delta I = 0$ mass differences assuming that

$$\epsilon \equiv \frac{1}{N} \sim \frac{1}{N_F} \sim \frac{m_s}{\Lambda}. \tag{2.3}$$

The predictions we obtain will be viewed as predictions for $\Delta S \sim 1$ mass differences, which are $O(\epsilon)$ at leading order in this expansion.

For the $\Delta I \neq 0$ mass differences, we assume that

$$\alpha_{\mathrm{EM}} \ll \frac{m_d - m_u}{\Lambda}. \tag{2.4}$$

To see that this is reasonable, we note that (to first order in the expansion performed below)

$$\frac{\alpha_{\mathrm{EM}}}{(m_d - m_u)/\Lambda} \sim \frac{m_{\pi^\pm}^2 - m_{\pi^0}^2}{m_{K^0}^2 - m_{K^\pm}^2} = 30\%, \tag{2.5}$$

$$\sim \frac{M_{\Sigma^+} - 2M_{\Sigma^0} + M_{\Sigma^-}}{M_{\Sigma^+} - M_{\Sigma^-}} = 20\%. \tag{2.6}$$

We therefore expand $\Delta I \neq 0$ mass differences to order $(m_d - m_u)\epsilon$ and α_{EM}. Except for the fact that we treat N as a large parameter, this expansion is identical to the second-order chiral expansion that is usually adopted in chiral perturbation theory [12,13].

D. Effective lagrangian

The effective lagrangian we use is described in ref. [6], and we will not review it in detail here. We keep track of the large-N group theory by writing the baryon fields $|B)$ as elements in a spin–flavor Fock space. The operators that couple to these fields are written as r-body operators in spin–flavor space with large-N (and N_F) suppression factors $1/N^{r+t-1}$, where t is the number of flavor traces used to write the operator [6,7]. The fields for the light pseudoscalar mesons Π are collected in the usual combination $\xi \equiv e^{i\Pi/f}$, where $f \simeq 114$ MeV is the kaon decay constant. (Note that the η' mass is of order N_F/N, and is therefore not included as a light field.) Explicit flavor breaking appears through the quark masses and electromagnetic charge matrix

$$m_q = \begin{pmatrix} m_u \mathbf{1}_{N_u} & & \\ & m_d \mathbf{1}_{N_d} & \\ & & m_s \mathbf{1}_{N_s} \end{pmatrix}, \qquad Q_q = \begin{pmatrix} \frac{2}{3}\mathbf{1}_{N_u} & & \\ & -\frac{1}{3}\mathbf{1}_{N_d} & \\ & & -\frac{1}{3}\mathbf{1}_{N_s} \end{pmatrix}. \tag{2.7}$$

E. Power counting for chiral loops

In this subsection, we discuss the power counting of loop graphs in the expansion described above. This power counting is more complicated than for ordinary chiral perturbation theory, and it is essential in order to ensure that we have not omitted any important contributions at the order we are working. (For example, we will see that there are 2-loop graphs that potentially contribute to the baryon mass differences at order $m_s^2 \ln m_s$, and we resort to an explicit calculation to see that it does not occur.) We begin by reviewing the power counting for "pure" baryon chiral perturbation theory (including only the lowest-lying baryon octet). We then discuss the new features that are present in the $1/N$ expansion.

In "pure" baryon chiral perturbation theory, a generic term in the effective lagrangian looks like [16]

$$\mathcal{L} \sim \sum \Lambda^2 f^2 \left(\frac{\partial}{\Lambda}\right)^d \left(\frac{m_q}{\Lambda}\right)^s \left(\frac{\Pi}{f}\right)^k \left(\frac{B}{f\sqrt{\Lambda}}\right)^n \tag{2.8}$$

where B is the baryon field, and Π is the meson field. (For brevity, we count $\alpha_{\mathrm{EM}} Q_q^2$ as a power of m_q in this subsection.) A loop diagram will contribute to the baryon mass

$$\delta M \sim \left(\frac{1}{16\pi^2 f^2}\right)^L \frac{1}{\Lambda^{D-2V_\Pi - V_B}} \left(\frac{m_q}{\Lambda}\right)^S m_\Pi^{2L+D-2V_\Pi - V_B+1} F(\Delta M/m_\Pi) \tag{2.9}$$

$$\sim \Lambda \left(\frac{\Lambda^2}{16\pi^2 f^2}\right)^L \left(\frac{m_q}{\Lambda}\right)^C F(\Delta M/m_\Pi) \tag{2.10}$$

where

$$C \equiv L + \tfrac{1}{2} + (\tfrac{1}{2} D_\Pi + S_\Pi - V_\Pi) + \tfrac{1}{2}(D_B + S_B - V_B) + \tfrac{1}{2} S_B. \tag{2.11}$$

Here, L is the number of loops, D_Π (D_B) is the total number of derivatives in meson (baryon) vertices, S_Π (S_B) is the total number of powers of m_q summed over the meson (baryon) vertices, V_Π (V_B) is the total number of pion (baryon) vertices, and $F(\Delta M/m_\Pi)$ is a function of the baryon mass differences and pion masses that appear in the loop diagram. Because $\Delta M/m_\Pi \sim m_q^{1/2}$, F can be expanded in a power series in $m_q^{1/2}$. The chiral suppression factor C is written in this form because $D_B + S_B - V_B$ and $\frac{1}{2} D_\Pi + S_\Pi - V_\Pi$ measure the total number number of "extra" insertions of derivatives and/or powers of m_q in baryon and meson vertices, respectively. This is because each baryon vertex contains at least one derivative or one power of m_q, and each each pion vertex contains at least two derivatives or one power of m_q.

As an application of this formula, note that at order $m_q^2 \ln m_q$ we must include 1-loop graphs with a single insertion of a 2-derivative baryon vertex ($L = 1$, $S_B = 0$, and $D_B + S_B - V_B = 1$). Since the 2-derivative meson vertices are not measured, we cannot compute the $m_q^2 \ln m_q$ logarithmic corrections to the baryon masses in baryon chiral perturbation theory [14].

We now turn to loop graphs in large-N baryon chiral perturbation theory. The chiral suppressions are as given above, and so we will concentrate on the N-dependence of diagrams. An arbitrary diagram naïvely gives a contribution to the baryon mass

$$\delta M \overset{?}{\lesssim} \Lambda \left(\frac{\Lambda^2}{16\pi^2 f^2}\right)^L \left(\frac{m_q}{\Lambda}\right)^C N^{V_B} F(\Delta M/m_\Pi). \tag{2.12}$$

Because a meson vertex can change the baryon flavor quantum numbers only by order 1 (rather than N), ΔM can have contributions that are at most $1/N$ or m_q. In our expansion $F(\Delta M/m_\Pi)$ can therefore be expanded in a power series in $1/N$ and $m_q^{1/2}$.

Since $f \sim N^{1/2}$, eq. (2.12) apparently violates large-N consistency if $V_B - L > 1$. However, there are cancellations among graphs that make the contributions to the physical mass differences at most order N. Actually, it has never been proven that the required cancellations occur to all orders, but many explicit calculations have been done that confirm this assertion [3,4].[2] The 2-loop calculation done in the appendix of this paper provides an additional highly nontrivial check.[3]

[2]In fact, the first few terms in the $1/N$ expansion for a given matrix element can be derived by demanding that these cancellations occur [2,3].

[3]2-loop corrections to the axial currents are considered in ref. [17].

In our expansion, the argument of F is

$$\frac{\Delta M}{m_\Pi} \sim \frac{1}{N} \left(\frac{\Lambda}{m_q}\right)^{1/2}, \tag{2.13}$$

and we can expand F in a power series in $\Delta M/m_\Pi$ for loops of K's and η's. (Loops with π intermediate states are negligible at the order we are working.) The terms with extra powers of $1/N$ arising from the expansion of F require fewer cancellations for consistency, and we obtain

$$\delta M \lesssim \Lambda \left(\frac{N\Lambda^2}{16\pi^2 f^2}\right)^L \left(\frac{m_q}{\Lambda}\right)^C \left[N^{\min\{V_B-L,1\}} + N^{\min\{V_B-L-1,1\}} \left(\frac{\Lambda}{m_q}\right)^{1/2} + \cdots\right]. \tag{2.14}$$

The leading contribution to the terms in the square brackets proportional to $m_q^{n/2}$ arise from diagrams with n insertions of the $1/N$-suppressed mass term Δ_0. The largest term in the brackets in eq. (2.14) is

$$N \left(\frac{\Lambda}{m_q}\right)^{(V_B-L-1)/2}, \tag{2.15}$$

so that

$$\delta M_B \leq N\Lambda \left(\frac{m_q}{\Lambda}\right)^{C-(V_B-L-1)/2}. \tag{2.16}$$

Using $V_B - L \leq L$, we can write

$$C - \tfrac{1}{2}(V_B - L - 1) \geq \tfrac{1}{2}(L+1) + (\tfrac{1}{2}D_\Pi + S_\Pi - V_\Pi) + \tfrac{1}{2}(D_B + S_B - V_B) + \tfrac{1}{2}S_B. \tag{2.17}$$

Using these results, we can enumerate all of the graphs that we need to consider to expand the baryon masses to second order in the expansion described above. From eq. (2.17) we see that we need to consider at most 2-loop graphs. Starting from the minimum number of insertions of Δ_0, it is easy to see that we need to consider the 1-loop graphs of fig. 2, the 1-loop graphs fig. 3 with zero or one insertion of Δ_0, and the 2-loop graphs of fig. 4 with one insertion of Δ_0. In the appendix it is shown that the leading contribution of the 2-loop graphs cancel, so there is no 2-loop contribution at this order.

III. EXPANSION OF BARYON MASSES

Using the formalism discussed above, we now turn to the expansion of the baryon masses.

A. Leading order

We first expand the baryon mass differences to order ϵ for the $\Delta I = 0$ mass differences and to order $\alpha_{\rm EM}$ and $(m_d - m_u)$ for the $\Delta I \neq 0$ mass differences. At this order, there are

no chiral loop corrections, and the expansion is determined by the following terms in the effective lagrangian:

$$\delta\mathcal{L} = -(B|\left[\Delta_0 + \Delta_1 + \cdots\right]|B), \tag{3.1}$$

where

$$\Delta_0 = \frac{\mu}{N}\{\sigma^j\}\{\sigma^j\}, \tag{3.2}$$

$$\Delta_1 = a_1\{m\} + \frac{\alpha_{\rm EM}\Lambda}{4\pi}\left[b_1\{Q\}\{Q\} + b_2\{Q\sigma^j\}\{Q\sigma^j\}\right]. \tag{3.3}$$

Here,

$$m \equiv \frac{1}{2}\left(\xi^\dagger m_q \xi + \text{h.c.}\right), \quad Q \equiv \frac{1}{2}\left(\xi^\dagger Q_q \xi^\dagger + \text{h.c.}\right). \tag{3.4}$$

Note that the coefficients of the operators $\{Q\}\{Q\}$ and $\{Q\sigma^j\}\{Q\sigma^j\}$ appear to violate the large-N counting rule discussed above. The reason is that at the quark level these operators arise from electromagnetic diagrams such as the one in fig. 5, which are not suppressed by a factor of $1/N$ from gluon vertices. However, in order to obtain a good large-N limit, we must demand that $\alpha_{\rm EM} \lesssim 1/N$ so that the electromagnetic Coulomb energy does not overwhelm the strong binding energy for large N. This also explains why the operator $\{Q^2\}$ is not considered at this order, since the coefficient of this operator is the same order as the electromagnetic operators considered above.[4] At this order we obtain the $\Delta I = 0$ mass relations

$$(\Xi - \Sigma) - (\Sigma - N) + \tfrac{3}{2}(\Sigma - \Lambda) = 0, \qquad (8\%) \tag{3.5}$$

$$(\Xi^* - \Sigma^*) - (\Sigma^* - \Delta) = 0, \qquad (3\%) \tag{3.6}$$

$$(\Omega - \Xi^*) - (\Xi^* - \Sigma^*) = 0, \qquad (7\%) \tag{3.7}$$

$$(\Sigma - N) - (\Lambda - N) = 0, \qquad (40\%) \tag{3.8}$$

$$(\Sigma^* - \Delta) - (\Lambda - N) = 0, \qquad (15\%) \tag{3.9}$$

and the $\Delta I \neq 0$ mass relations

[4] In fact, this is a good example of how the $1/N$ expansion can differ qualitatively from the nonrelativistic quark model. In the quark model, the operators $\{Q\}\{Q\}$ and $\{Q^2\}$ are expected to be of the same order, while the spin-symmetry violating operator $\{Q\sigma^j\}\{Q\sigma^j\}$ would be suppressed by $1/m_Q^2$, where m_Q is the constituent quark mass. In the constituent quark model, we obtain the additional relation $\Sigma^{*-} - \Sigma^{*+} = \Sigma^- - \Sigma^+$ with accuracy $40 \pm 15\%$. Unfortunately, this is neither sufficiently successful nor sufficiently unsuccessful to decide whether the quark model works better than the $1/N$ expansion discussed here.

$$(\Xi^- - \Xi^0) - (\Sigma^- - \Sigma^+) + (n - p) = 0, \qquad (7 \pm 11\%) \tag{3.10}$$

$$\Delta^{++} - 3\Delta^+ + 3\Delta^0 - \Delta^- = 0, \tag{3.11}$$

$$(\Xi^{*-} - \Xi^{*0}) - (\Sigma^{*-} - \Sigma^{*+}) + (n - p) = 0, \qquad (2 \pm 22\%) \tag{3.12}$$

$$(\Sigma^{*+} - 2\Sigma^{*0} + \Sigma^{*-}) - (\Sigma^+ - 2\Sigma^0 + \Sigma^-) = 0, \qquad (40 \pm 100\%) \tag{3.13}$$

$$(\Delta^0 - \Delta^+) - (n - p) = 0, \tag{3.14}$$

$$(\Sigma^{*-} - \Sigma^{*+}) - (\Sigma^+ - 2\Sigma^0 + \Sigma^-) - 2(n - p) = 0, \qquad (3 \pm 7\%) \tag{3.15}$$

$$(\Delta^0 - \Delta^{++}) + (\Sigma^{*-} - \Sigma^{*+}) - 4(n - p) = 0, \qquad (40 \pm 20\%) \tag{3.16}$$

In addition, we can extract the quark mass ratio

$$R \equiv \frac{m_s - (m_u + m_d)/2}{m_d - m_u} = 6\frac{\Lambda - N}{(\Sigma^+ - \Sigma^-) - 4(\Sigma^{*+} - \Sigma^{*-})} = 110 \pm 30. \tag{3.17}$$

Here, the name of a baryon denotes its mass. Among the $\Delta I = 0$ mass relations, eq. (3.5) (the Gell-Mann–Okubo relation) and eqs. (3.6) and (3.7) (the decuplet equal spacing rules) are valid at order m_s independently of the $1/N$ expansion, while eqs. (3.8) and (3.9) are consequences of the $1/N$ expansion. (In this limit, the mass of a baryon with strangeness $-S$ is proportional to S.) Among the $\Delta I \neq 0$ mass relations, eq. (3.10) (the Coleman–Glashow relation) is valid up to corrections of order $(m_d - m_u)m_s$ independently of the $1/N$ expansion [14]. Also, the only corrections to eq. (3.11) are second order in isospin breaking independently of the $1/N$ expansion. The remaining $\Delta I \neq 0$ relations eqs. (3.12)–(3.16) as well as eq. (3.17) are consequences of the $1/N$ expansion.

The relations are written as linear combinations of mass differences that go to zero as $m_s \to 0$ (for the $\Delta I = 0$ relations) or $(m_d - m_u) \to 0$, $\alpha_{\rm EM} \to 0$ (for the $\Delta I \neq 0$ relations). The accuracy quoted for the relations (where data is available) is defined as the deviation from zero divided by the average of the absolute value of the mass differences that appear in the equation. Defined in this way, all of these relations are naïvely expected to have corrections of order $\epsilon \sim 30\%$, except for eq. (3.11), which is essentially exact in our expansion. Note that for the $\Delta I \neq 0$ mass differences the relations that hold as a consequence of flavor symmetry alone work better than those that depend on the $1/N$ expansion, but there is apparently no such pattern for the $\Delta I \neq 0$ differences.

The value for R obtained is far from the value $R \simeq 25$ obtained from an analysis of the light pseudoscalar masses [12]. The quoted error only takes into account the experimental uncertainty of the masses and does not include the theoretical uncertainty from higher-order corrections. In view of the large errors in the relations above, we do not take this value very seriously.

A more objective measure of how well these relations work is obtained by fitting the mass differences to the parameters given above. The average deviation in a the best fit is 30 MeV for the $\Delta I \neq 0$ mass differences and 0.35 MeV for the $\Delta I \neq 0$ differences. (If we omit the model-dependent value for $\Delta^0 - \Delta^{++}$, we get an average deviation of 0.20 MeV for the $\Delta I \neq 0$ mass differences.) The fit also gives $R \simeq 90$.

B. Chiral loops

The largest corrections to the leading order results in the expansion we are performing come from the loop diagrams in fig. 3 and are of order $Nm_s^{3/2}$ for the $\Delta I = 0$ mass differences and order $Nm_s^{1/2}(m_d - m_u)$ for the $\Delta I \neq 0$ mass differences. These diagrams can be evaluated from

$$\delta M_B = (B|\{T_A\sigma^j\}\{T_A\sigma^k\}|B)\,\frac{ig^2}{6f^2}\int\frac{d^4p}{(2\pi)^4}\,\frac{p^j p^k}{(p^2 - m_A^2)\,p_0}, \tag{3.18}$$

where $A = K, \eta$, *etc.* labels the pseudoscalar mass eigenstates and T_A are the corresponding generators normalized so that $\mathrm{tr}(T_A T_B) = \delta_{AB}$. The effects of π^0–η mixing are incorporated by using the generator

$$T_\eta = T_8 - \frac{(N_u N_d N_s N_F)^{1/2}}{N_u + N_d}\,\frac{1}{2R}T_3 + O(1/R^2) \tag{3.19}$$

for the η mass eigenstate, where R is the quark mass ratio defined in eq. (3.17). The result can be written

$$\delta M_B = (B|J_A \mathcal{O}_A|B), \qquad J_A = \frac{g^2}{16\pi f^2}m_A^3, \tag{3.20}$$

where

$$\mathcal{O}_K = -(N + N_F - 2N_s)\{S\} + \frac{1}{3}\{S\sigma^j\}\{\sigma^j\} + \{S\}\{S\} - \frac{1}{3}\{S\sigma^j\}\{S\sigma^j\}, \tag{3.21}$$

$$\begin{aligned}\mathcal{O}_\eta = {}& \frac{2}{3(N_F - N_s)}\{S\sigma^j\}\{\sigma^j\} - \frac{N_F}{3N_s(N_F - N_s)}\{S\sigma^j\}\{S\sigma^j\} \\ &+ \frac{1}{6R}\left[\frac{N_s}{N_u + N_d}\{\tau_3\sigma^j\}\{\sigma^j\} + \{\tau_3\sigma^j\}\{S\sigma^j\}\right],\end{aligned} \tag{3.22}$$

$$\mathcal{O}_{K^\pm} - \mathcal{O}_{K^0} = -\frac{N_s}{2}\{\tau_3\} - \frac{1}{2}\{\tau_3\}\{S\} + \frac{1}{6}\{\tau_3\sigma^j\}\{S\sigma^j\}. \tag{3.23}$$

Here,

$$S \equiv \begin{pmatrix} \mathbf{0}_{N_u} & & \\ & \mathbf{0}_{N_d} & \\ & & \mathbf{1}_{N_s} \end{pmatrix}, \qquad \tau_3 \equiv \frac{2}{N_u + N_d}\begin{pmatrix} N_d\mathbf{1}_{N_u} & & \\ & -N_u\mathbf{1}_{N_d} & \\ & & \mathbf{0}_{Ns} \end{pmatrix}. \tag{3.24}$$

We do not need expressions for the pion loops, since they are suppressed by $\sim (m_{u,d}/m_s)^{3/2}$ for $\Delta I = 0$ quantities, and by $\sim \alpha_{\mathrm{EM}}\Lambda/m_{u,d}$ for $\Delta I \neq 0$ quantities (since the contribution of pion loops is proportional to $m_{\pi^+} - m_{\pi^0}$, which is purely electromagnetic at first order in chiral perturbation theory).

Note that eqs. (3.21) through (3.23) are valid for arbitrary $N_{u,d,s}$ and N_F, and that they have a sensible limit as $N, N_F \to \infty$ independently of the extrapolation of the quantities $N_{u,d,s}$. According to the rules of our expansion, we must keep the full dependence on $N_{u,d,s}$

and N_F since there are limits where each of the above terms is important. The physical results are obtained by simply setting $N_F = 3$, $N_{u,d,s} = 1$.[5]

Substituting these expressions into the lowest order relations, we obtain the modified $\Delta I = 0$ relations

$$\begin{aligned}
(\Xi - \Sigma) - (\Sigma - N) + \tfrac{3}{2}(\Sigma - \Lambda) &= \tfrac{4}{3} J_K - J_\eta \\
&= -2 \pm 1 \text{ MeV}, &(3.25)\\
(\Xi^* - \Sigma^*) - (\Sigma^* - \Delta) &= \tfrac{4}{3} J_K - J_\eta \\
&= -2 \pm 1 \text{ MeV}, &(3.26)\\
(\Omega - \Xi^*) - (\Xi^* - \Sigma^*) &= \tfrac{4}{3} J_K - J_\eta \\
&= -2 \pm 1 \text{ MeV}, &(3.27)\\
(\Sigma - N) - (\Lambda - N) &= -\tfrac{4}{3}(J_K + J_\eta) \\
&= -400 \pm 160 \text{ MeV} &(3.28)\\
(\Sigma^* - \Delta) - (\Lambda - N) &= \tfrac{2}{3}(J_K + J_\eta) \\
&= 200 \pm 80 \text{ MeV} &(3.29)
\end{aligned}$$

and the modified $\Delta I \neq 0$ relations

$$\begin{aligned}
(\Sigma^{*-} - \Sigma^{*+}) - (\Sigma^+ - 2\Sigma^0 + \Sigma^-) - 2(n - p) &= -\frac{4}{3}(J_{K^0} - J_{K^+}) + \frac{2}{3R} J_\eta \\
&= (-4.0 \pm 1.6) + (4.8 \pm 1.9)\left(\frac{24}{R}\right) \quad (0.1 \pm 0.6 \text{ MeV}) &(3.30)\\
(\Delta^0 - \Delta^{++}) + (\Sigma^{*-} - \Sigma^{*+}) - 4(n - p) &= -\frac{4}{3}(J_{K^0} - J_{K^+}) + \frac{2}{3R} J_\eta \\
&= (-4.0 \pm 1.6) + (4.8 \pm 1.9)\left(\frac{24}{R}\right) \quad (1.6 \pm 0.85 \text{ MeV}) &(3.31)
\end{aligned}$$

where the errors in the theoretical prediction are obtained by assigning a 20% uncertainty to the coupling $g = 0.8$ extracted from a lowest-order fit to the $\Delta S = 1$ semileptonic baryon decays, and the experimental values are shown in parenthesis. (The remaining $\Delta I \neq 0$ relations do not receive corrections at this order.) In the $\Delta I = 0$ relations, the $O(m_s^{3/2})$ corrections to the relations that follow from $SU(N_F)$ flavor symmetry alone are small due to an "accidental" cancellation. (This was noted in ref. [15].) However, even taking into account the theoretical errors, the corrections to the relations that follow from the large-N expansion are too large and have the wrong sign. The same conclusion does not appear to hold for the corrections to the $\Delta I \neq 0$ relations, although the situation is obscured by the large uncertainties involved. This situation is analogous to what happens for the magnetic

[5]Note that for $N_F > 3$ there are "extra" pseudoscalar mesons transforming in the fundamental representation of $SU(N_q)$ that also contribute in the chiral loops. The masses of these states are determined in terms of the "physical" states by $SU(N_F)$ flavor symmetry, and so these contributions are calculable. However, these contributions go to zero in the physical limit $N_F = 3$, and so there is no reason to compute them explicitly.

moments, where the $O(m_s^{1/2})$ corrections to the large-N relations are too large, although of the right sign [8].

One can view this situation in several ways. The most conservative view is that these results show that the expansion we are performing does not work well. We will instead adopt the attitude that the apparently large corrections at this order of the expansion may be misleading, and go on to higher orders of the expansion. At higher orders, these large loop contributions can be cancelled by counterterms, and we will see that the predictions work very well.

To see why our viewpoint may be reasonable, we note that there are several calculable higher-order effects that we have omitted in our calculation, all of which substantially suppress the correction. The first of these are higher-order corrections to the meson coupling g. These are known to be large and apparently reduce g [16,18]. Second, we have checked that using the exact kinematics for the particles in the loops decreases the loop corrections by approximately a factor of $\frac{1}{2}$ for the large-N relations. Third, we expect that the meson–baryon coupling decreases at large momentum transfer, reducing the effects of K and η loops. All of these effects are suppressed by additional powers of m_s, and are therefore higher order in the expansion we are performing. Although none of these corrections changes the sign of the loop corrections, it is possible that these suppressions are large enough that the expansion we are performing makes sense at higher orders. We certainly cannot prove conclusively that this point of view is correct, but we can obtain evidence for it by going to higher orders.

There are also nonanalytic corrections of order $m_q^2 \ln m_q$. These are formally larger than the analytic corrections due to the counterterms by $\ln \Lambda^2 / M_K^2 = 1.4$ for $\Lambda = 1$ GeV. While these effects are definitely enhanced for m_s sufficiently small, we believe that for the physical value of m_s this enhancement is not numerically large enough to give reliable predictions without including the analytic counterterms.

C. Second order

In order to expand the $\Delta I = 0$ mass differences to order ϵ^2 and the $\Delta I \neq 0$ mass differences to order α_{EM} and $(m_d - m_u)\epsilon$, we must include the additional terms

$$\Delta_2 = \frac{a_2}{N}\{m\sigma^j\}\{\sigma^j\} + \frac{a_3}{\Lambda}\{m^2\} + \frac{a_4}{N\Lambda}\{m\}\{m\} + \frac{a_5}{N\Lambda}\{m\sigma^j\}\{m\sigma^j\}. \tag{3.32}$$

In addition to the analytic contributions, there are nonanalytic contributions from loop graphs of order $m_q^2 \ln m_q$ and $(m_q \ln m_q)/N$. There are $m_q^2 \ln m_q$ contributions from graphs such as fig. 2 with the baryon–meson vertices coming from higher-order operators such as

$$\frac{1}{\Lambda}\{A^j A^j\}, \quad \frac{1}{N\Lambda}\{A^j\}\{A^j\}, \quad \frac{1}{N\Lambda}\{A^j\sigma^j\}\{A^k\sigma^k\}. \tag{3.33}$$

There are also $m_q^2 \ln m_q$ contributions from graphs such as fig. 3 with one of the baryon–meson vertices coming from the higher-order operators

$$\frac{1}{\Lambda}\{(\nabla_0 A^j)\sigma^j\}, \quad \frac{1}{\Lambda}\{(\nabla_j A^0)\sigma^j\}. \tag{3.34}$$

The coupling constants associated with these operators are not well-measured, and we will treat them as free parameters. These graphs give contributions to the baryons masses of the form

$$\delta M_B = (B|C_j \mathcal{O}_{jA} K_A|B), \qquad K_A \equiv \frac{1}{16\pi^2 f^2 \Lambda} m_A^4 \ln \frac{\Lambda^2}{m_A^2}, \tag{3.35}$$

where the C_j are unknown constants and the independent operators that contribute at this order are

$$\mathcal{O}_{1A} = \{T_A\}\{T_A\}, \qquad \mathcal{O}_{2A} = \{T_A \sigma^j\}\{T_A \sigma^j\}. \tag{3.36}$$

There are also $(m_q \ln m_q)/N$ corrections arising from graphs such as fig. 3 with a single insertion of the flavor-independent baryon mass operator $\Delta_0 = \mu\{\sigma^j\}\{\sigma^j\}/N$. These contributions are proportional to

$$\begin{aligned} &\frac{g^2}{16\pi^2 f^2} [\{T_A \sigma^j\}, [\{T_A \sigma^j\}, \Delta_0]]\, m_A^2 \ln \frac{\Lambda^2}{m_A^2} \\ &\qquad \propto \frac{g^2}{16\pi^2 f^2} \frac{\mu}{N} \left[\{T_A \sigma^j\}\{T_A \sigma^j\} - \{T_A T_A \sigma^j\}\{\sigma^j\} \right] m_A^2 \ln \frac{\Lambda^2}{m_A^2} \qquad (3.37) \\ &\qquad \sim \frac{1}{N^2} \left[N^2 + N N_F \right] m_q \ln m_q. \qquad (3.38) \end{aligned}$$

The normalization of these contributions is calculable, but will not be needed.

When we expand the operators above in terms of the $SU(N_F)$-violating spurions S and τ_3, we see that the operators that appear in $\mathcal{O}_1$ and $\mathcal{O}_2$ are linear combinations of the operators that appear in the tree terms. However, the loop contributions are important because they eliminate a prediction for the quark mass ratio R that would otherwise exist at this order.

Eliminating the unknown constants leads to the relations

$$\begin{aligned} (\Omega - \Xi^*) - 2(\Xi^* - \Sigma^*) + (\Sigma^* - \Delta) &= 0, \qquad (1\%) \qquad (3.39) \\ (\Xi^* - \Sigma^*) - (\Sigma^* - \Delta) - (\Xi - \Sigma) + (\Sigma - N) - \tfrac{3}{2}(\Sigma - \Lambda) &= 0, \qquad (6\%) \qquad (3.40) \\ (\Xi^* - \Sigma^*) - (\Xi - \Sigma) &= 0, \qquad (17\%) \qquad (3.41) \\ (\Xi^- - \Xi^0) - (\Sigma^- - \Sigma^+) + (n - p) &= 0, \qquad (7 \pm 11\%) \qquad (3.42) \\ \Delta^{++} - 3\Delta^+ + 3\Delta^0 - \Delta^- &= 0, \qquad (3.43) \\ (\Xi^{*-} - \Xi^{*0}) - (\Sigma^{*-} - \Sigma^{*+}) + (n - p) &= 0, \qquad (2 \pm 22\%) \qquad (3.44) \\ (\Sigma^{*+} - 2\Sigma^{*0} + \Sigma^{*-}) - (\Sigma^+ - 2\Sigma^0 + \Sigma^-) &= 0, \qquad (40 \pm 100\%) \qquad (3.45) \\ (\Delta^0 - \Delta^+) - (n - p) &= 0, \qquad (3.46) \\ (\Delta^0 - \Delta^{++}) + (\Sigma^+ - 2\Sigma^0 + \Sigma^-) - 2(n - p) &= 0, \qquad (70 \pm 30\%) \qquad (3.47) \end{aligned}$$

Of the $\Delta I = 0$ relations, eq. (3.39) is an improved version of the equal spacing rule that holds to order m_s^2 independently of the $1/N$ expansion [15] while eqs. (3.40) and (3.41) are non-trivial predictions of the $1/N$ expansion. These relations were derived in this expansion in ref. [7]; the same relations are derived in a different expansion in ref. [4]. Of the $\Delta I \neq 0$

relations, eqs. (3.42) through (3.46) are identical to the lowest-order relations eqs. (3.10) through (3.14). It is worth noting that the Coleman–Glashow relation eq. (3.42) receives calculable *analytic* corrections of order $(m_d - m_u)m_s$ in chiral perturbation theory [14]; the results above show that these corrections are suppressed by $1/N$.

A fit to the measured mass differences gives an average deviation of 7 MeV for the $\Delta I = 0$ mass differences (compared to 29 MeV at lowest order). For the $\Delta I \neq 0$ mass differences, the average deviation is 0.35 MeV, or 0.20 MeV if we exclude the Δ^0–Δ^{++} mass difference. Since the average of the $\Delta I = 0$ mass differences is approximately 150 MeV and the average of the $\Delta I \neq 0$ mass differences is approximately 3.5 MeV, this is consistent with the expected accuracy of $1/N^2 \sim 10\%$. Note that the result for the $\Delta I \neq 0$ mass difference is the same as at first order, but this simply reflects the fact that the first-order relations work better than expected.

D. Higher orders

At next order in large-N chiral perturbation theory, we must include the operators

$$\begin{aligned}\Delta_3 = &\frac{a_6}{N^2}\{m\}\{\sigma^j\}\{\sigma^j\} + \frac{a_7}{N^2\Lambda}\{m\}\{m\sigma^j\}\{\sigma^j\},\\ &+ \frac{a_8}{N^2\Lambda^2}\{m\}\{m\}\{m\} + \frac{a_9}{N^2\Lambda^2}\{m\}\{m\sigma^j\}\{m\sigma^j\}\end{aligned} \tag{3.48}$$

which give contributions to the mass differences of order m_q/N^2, m_q^2/N, and m_q^3. Including these terms in addition to the second-order effects discussed above, there is one surviving relation:

$$(\Sigma^{*+} - 2\Sigma^{*0} + \Sigma^{*-}) = (\Sigma^+ - 2\Sigma^0 + \Sigma^-). \qquad (40 \pm 100\%) \tag{3.49}$$

This relation gets corrections from the term

$$\frac{\alpha_{\rm EM}\Lambda}{4\pi}\,\frac{1}{N\Lambda}\{Q\}\{Q\sigma^j\}\{m\sigma^j\} \tag{3.50}$$

at order $\alpha_{\rm EM} m_s$, and also possibly from loop effects at order $m_q^{5/2}$, $m_q^3 \ln m_q$, *etc.* (Checking this requires us to consider 3-loop graphs.) In view of the large experimental uncertainty in this relation, we will not pursue the expansion beyond this point.

IV. CONCLUSIONS

We have analyzed the baryon mass differences in large-N baryon chiral perturbation theory with particular emphasis on the chiral loop corrections. One result of our work is that the nonanalytic $m_s^{3/2}$ corrections to the $\Delta I = 0$ large-N mass relations appear to be too large and have the wrong sign to explain the corrections to the lowest-order relations. Nonetheless, accurate results are obtained at higher order in this expansion. At second order, there are eight non-trivial predictions of the $1/N$ expansion (which do not follow

from flavor symmetry alone). These relations work to better than 10% accuracy, consistent with the assumption that the errors are $\sim 1/N^2 \sim m_s^2/\Lambda^2$.

Ref. [10] also analyzes the baryon masses in a combined expansion in $1/N$ and flavor breaking. The main differences from the present work is that ref. [10] performs a different expansion in which only baryons with strangeness of order 1 in the large-N limit are considered, and nonanalytic chiral loop corrections are not included. Loop corrections in the expansion of ref. [10] are considered in ref. [19].

APPENDIX: BARYON MASSES TO TWO LOOPS

In this Appendix, we compute the 2-loop contributions to the baryon masses shown in fig. 4. (The other two-loop graphs can be seen to be negligible using the power-counting arguments of section IIIB.) We will find that these contributions are negligible, but we do not know any simpler way to see this than by computing them explicitly.

General Formula

We begin by deriving the general formula for the 2-loop contribution to the masses in the presence of mixing. The baryon self-energy can be viewed as an operator $\Gamma(E)$ in the spin-flavor Fock space, where E is the energy of the baryon. If we denote the physical baryon fields by $|n)$, the mass eigenvalues are determined by

$$\Gamma(E_n)|n) = 0. \tag{4.1}$$

This is a nonlinear eigenvalue equation to be solved simultaneously for E_n and $|n)$ order by order in the loop expansion. The first step is to expand all quantities in the number of loops:

$$\begin{aligned} \Gamma(E_n) &= E_n - \Delta - L_1(E_n) - L_2(E_n) + \cdots, & (4.2)\\ E_n &= E_n^{(0)} + E_n^{(1)} + E_n^{(2)} + \cdots, & (4.3)\\ |n) &= |n^{(0)}) + |n^{(1)}) + |n^{(2)}) + \cdots. & (4.4) \end{aligned}$$

The energy dependence of the loop perturbations must also be expanded:

$$L_j(E_n) = L_j(E_n^{(0)}) + L_j'(E_n^{(0)})E_n^{(1)} + \cdots. \tag{4.5}$$

Writing out eq. (4.1) and equating terms at the same order in the loop expansion gives at tree level

$$\Delta|n^{(0)}) = E_n^{(0)}|n^{(0)}), \tag{4.6}$$

at one loop[6]

[6] We are assuming that all degeneracies are lifted by the tree-level terms, so that we do not need degenerate perturbation theory.

$$E_n^{(1)} = (n^{(0)}|L_1(E_n^{(0)})|n^{(0)}), \tag{4.7}$$

$$|n^{(1)}) = \sum_{m\neq n} |m^{(0)}) \frac{(m^{(0)}|L_1(E_n^{(0)})|n^{(0)})}{E_n^{(0)} - E_m^{(0)}}, \tag{4.8}$$

and at two loops

$$E_n^{(2)} = (n^{(0)}|L_2(E_n^{(0)})|n^{(0)}) + E_n^{(1)}(n^{(0)}|L_1'(E_n^{(0)})|n^{(0)}) + \sum_{m\neq n} \frac{|(m^{(0)}|L_1(E_n^{(0)})|n^{(0)})|^2}{E_n^{(0)} - E_m^{(0)}}. \tag{4.9}$$

The "energy denominator" terms in the two-loop formula are irrelevant in the limit where we can ignore mixing,

$$(m^{(0)}|L_1(E_n^{(0)})|n^{(0)}) = 0 \quad \text{for } m \neq n. \tag{4.10}$$

For the problem at hand, mixing always violates isospin, so that the energy denominator terms in eq. (4.9) are second order in isospin violation. This is true for arbitrary N and N_F, as can be seen by considering the generalized isospin and strangeness discussed in section IIB as good quantum numbers. Since the baryons of given spin J form an irreducible representation of $SU(N_F)$, and because any state in such a representation is uniquely specified by these quantum numbers, we see that mixing always violates generalized isospin. We can then write

$$E_n^{(2)} = (n^{(0)}|L_2(E_n^{(0)}) + \tfrac{1}{2}\left[L_1(E_n^{(0)}), L_1'(E_n^{(0)})\right]_+ |n^{(0)}). \tag{4.11}$$

This formula gives the precise meaning of the graphs in fig. 4. The anticommutator term can be thought of as arising from wavefunction renormalization.

2-loop Contribution

We begin by considering the "true" 2-loop graphs; we will consider the 1-loop graphs with counterterm insertions in the following subsection. By the power-counting considerations of subsection IIIB, we need only consider graphs of the form in fig. 4 with a single insertion of Δ_0.[7] These integrals are regulated using *e.g.* dimensional regularization so that all the manipulations below are well-defined. This contribution is a sum of terms with momentum integrals of the form

$$\int \frac{d^4p}{(2\pi)^4} \frac{p^j p^k}{p^2 - m_A^2} \int \frac{d^4q}{(2\pi)^4} \frac{q^\ell q^m}{q^2 - m_A^2} F(p_0, q_0) \propto \delta^{jk}\delta^{\ell m} \tag{4.12}$$

by 3-dimensional rotational invariance. We then obtain

[7]We have also evaluated the 2-loop graphs with no mass insertion and verified that the N dependence is consistent.

$$
\begin{aligned}
\delta M_B = \frac{g^4}{36 f^4} \int \frac{d^4 p}{(2\pi)^4} \frac{\vec{p}^{\,2}}{p^2 - m_A^2} \int \frac{d^4 q}{(2\pi)^4} \frac{\vec{q}^{\,2}}{q^2 - m_B^2} & \\
\Bigg[\frac{1}{p_0^3 (p_0 + q_0)} (AXBBA + ABBXA) & \\
+ \frac{1}{p_0^2 (p_0 + q_0)^2} ABXBA & \\
+ \frac{1}{p_0^2 q_0 (p_0 + q_0)} AXBAB & \\
+ \frac{1}{p_0 q_0 (p_0 + q_0)^2} ABXAB & \\
+ \frac{1}{p_0 q_0^2 (p_0 + q_0)} ABAXB & \\
- \left(\frac{1}{2 p_0^2 q_0^2} + \frac{1}{p_0^3 q_0} \right) (AXABB + BBAXA) \Bigg], & \qquad (4.13)
\end{aligned}
$$

where we have used the abbreviations

$$
A \equiv \{T_A \sigma^j\}, \quad B \equiv \{T_B \sigma^k\}, \qquad (4.14)
$$

and *e.g.*

$$
\begin{aligned}
ABXBA &\equiv AB\Delta_0 BA - \tfrac{1}{2} (\Delta_0 ABBA + ABBA\Delta_0) & (4.15)\\
&= \tfrac{1}{2} [AB, \Delta_0] BA + \tfrac{1}{2} AB [\Delta_0, BA], & (4.16)
\end{aligned}
$$

where $\Delta_0 = \mu \{\sigma^j\}\{\sigma^j\}/N$.

Eq. (4.13) is naively a 5-body operator (if we take into account the commutator structure defined by X insertions), but it must be a 4-body operator by large-N consistency. By explicit calculation, we find that the 5-body part of eq. (4.13) indeed vanishes, and we obtain

$$
\begin{aligned}
\delta M_B = \frac{g^4}{36 f^4} \int \frac{d^4 p}{(2\pi)^4} \frac{\vec{p}^{\,2}}{p^2 - m_A^2} \int \frac{d^4 q}{(2\pi)^4} \frac{\vec{q}^{\,2}}{q^2 - m_B^2} & \\
\Bigg(\frac{1}{2 p_0^2 q_0^2} AB[A, [B, \Delta]] & \\
+ \frac{4 q_0 + 3 p_0}{2 p_0^3 q_0 (p_0 + q_0)} AB[[A, B], \Delta] & \\
+ \frac{2 p_0^3 + 3 p_0^2 q_0 + 2 p_0 q_0^2 + 2 q_0^3}{p_0^3 q_0^3 (p_0 + q_0)} A[A, B][B, \Delta] \Bigg) & \\
+ \text{lower-body operators.} & \qquad (4.17)
\end{aligned}
$$

Evaluating the 4-body operators that appear, we find that they are all $\lesssim N^2$, and we have checked that all lower-body operators that appear are also $\lesssim N^2$ even for $N_F \sim N$. This shows that

$$
\delta M_B \lesssim N^0 m_q^2 \ln m_q, \qquad (4.18)
$$

which is negligible in our expansion.

Note that there is a 4-body operator that could have contributed, namely

$$\begin{aligned} AA[B,[B,\Delta]] &\sim \frac{1}{N} AA \left([B,\{\sigma^j\}][B,\{\sigma^j\}] + \{\sigma^j\}[B,[B,\{\sigma^j\}]] + \cdots \right) \\ &\sim \frac{1}{N} N^2 [N^2 + N_F N + \cdots]. \end{aligned} \tag{4.19}$$

Therefore, the important point of this calculation is that this operator does not appear in the evaluation of the 2-loop graphs.

One loop counterterm graphs

As long as we are evaluating 1-loop graphs, we can use dimensional regularization and minimal subtraction, which allow us to simply drop the divergent parts. When we consider 2-loop graphs, we must be more careful about the counterterm structure, since we must also evaluate the 1-loop graphs with a single insertion of a 1-loop counterterm.

As already stated above, diagrams with L loops generally scale as N^L for large N. This means that there are divergences in 1PI graphs that cannot be cancelled by counterterms with the same N-dependence as the tree-level lagrangian. Even at 1 loop, there is a divergent wavefunction renormalization of order N, while the tree-level kinetic term is order 1. This does not imply that the lagrangian is not closed under renormalization, since we can rescale the fields so that the counterterms have the correct form.[8] We therefore write the (bare) effective lagrangian as

$$\mathcal{L} = (B| Z^{1/2} \left[iv \cdot \nabla - \Delta + g\{A \cdot \sigma\} + \cdots + \mathcal{O}_{\rm ct} \right] Z^{1/2} |B), \tag{4.20}$$

where Z and $\mathcal{O}_{\rm cr}$ are divergent counterterms to be chosen order-by-order in the loop expansion to render all graphs finite. By allowing Z to have arbitrary N dependence, we can cancel the divergences with counterterms $\mathcal{O}_{\rm ct}$ that have the same form as the operators appearing in the tree-level lagrangian. This is the sense in which the bare lagrangian has the same form as the tree-level lagrangian.

At one loop, we find

$$Z = 1 - \frac{g^2}{6f^2} \{T_A \sigma^j\}\{T_A \sigma^j\} + O(\Delta^2) \tag{4.21}$$

$$\mathcal{O}_{\rm ct} = -\frac{g^2}{12 f^2} [\{T_A \sigma^j\}, [\{T_A \sigma^j\}, \Delta - g\{A \cdot \sigma\}]] \,\mathrm{div} \int \frac{d^4 p}{(2\pi)^4} \frac{\vec{p}^{\,2}}{p^2 - m_A^2} \frac{1}{p_0^2}. \tag{4.22}$$

Here, "div" indicates the divergent part.

The counterterms in $\mathcal{O}_{\rm ct}$ are smaller by one power of $1/N$ than required by large-N consistency. This can be used to see that the contribution of the 1-loop counterterm graphs

[8] Again, this has not been proven to all loops.

to the baryon masses are $\sim N^0$ (not $\sim N$). The reason is simply that the physical mass is independent of the scale of the fields, and so one can compute the masses with $Z = 1$. All the other counterterms that appear are now suppressed by an additional power of $1/N$, which is immediately gives the result that these contributions are negligible in our expansion.

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FIGURE CAPTIONS

FIG. 1: The Young tableaux for the $SU(N_F)$ flavor representation of the spin-J baryon multiplet.

FIG. 2: A contribution to the baryon mass. The solid lines are baryons and the dashed lines are mesons.

FIG. 3: A contribution to the baryon mass.

FIG. 4: Two-loop contributions to the baryon mass.

FIG. 5: Example of a quark graph giving rise to the 2-body operators $\{Q\}\{Q\}$ and $\{Q\sigma^j\}\{Q\sigma^j\}$ in the effective lagrangian. The wavy line represents a photon and the curly line represents a gluon. Note that there is a factor of N from the color sum, so this graph is order 1 in the large-N limit.

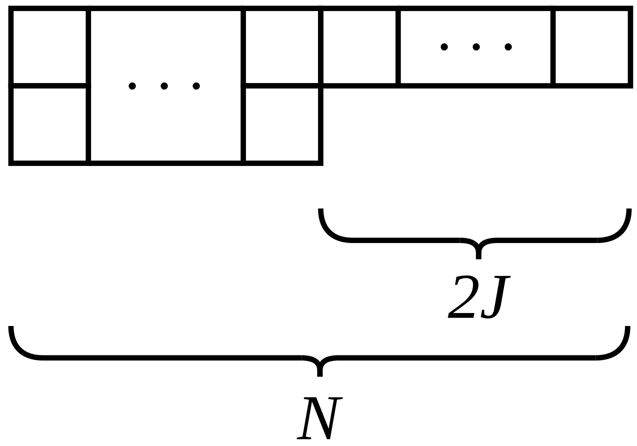

FIG. 1

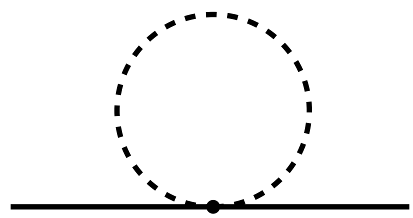

FIG. 2

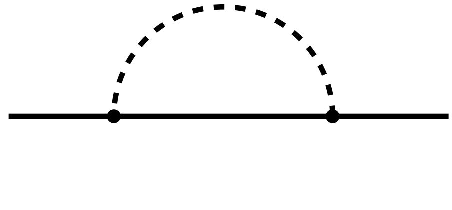

FIG. 3

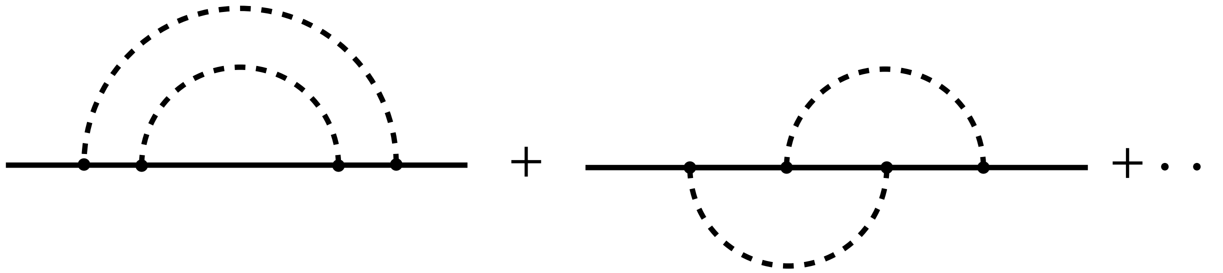

FIG. 4

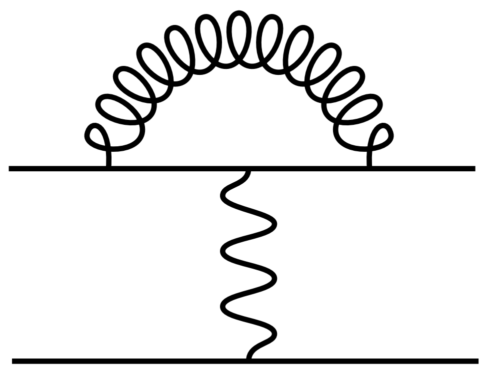

FIG. 5